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Propagation of Whittaker-Gaussian Beams

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ABSTRACT

We study the propagating and shaping characteristics of the novel Whittaker-Gaussian beams (WGB). The transverse profile is described by the Whittaker functions. Their physical characteristics are studied in detail by finding the 2n-order intensity moments of the beam. Propagation through complex ABCD optical systems, normalization factor, beamwidth, the quality M^2 factor and its kurtosis parameter are derived. We discuss its behavior for different beam parameters and the relation between them. The WGBs carry finite power and form a biorthogonal set of solutions of the paraxial wave equation (PWE) in circular cylindrical coordinates.

Keywords: Paraxial wave equation; Bessel-Gauss beams; Laguerre-Gauss beams; ABCD systems; Beam propagation factor.

1. INTRODUCTION

Recently, the Circular beams (CiBs) were introduced by Bandres and Gutiérrez-Vega[1] as the general solution of the paraxial wave equation in circular cylindrical coordinates. The complex amplitude of the CiB can be described by either the Whittaker functions or the confluent hypergeometric functions, and are characterized by four independent parameters. Besides the possibility of obtaining novel and meaningful beam structures, one of the most important aspects of the CiBs is that, for special values of their four parameters, they reduce to known families of optical beams including the standard, elegant, and generalized Laguerre-Gauss (LG) beams, the Bessel-Gauss beams, the Hypergeometric beams, Hypergeometric-Gauss beams, the fractional order elegant Laguerre-Gauss beams, the Bessel-Gauss beams with quadratic radial dependence, and the optical vortex beams. Finding general properties of the CiBs is a way to characterize all these special cases at the same time.

In this work, we study in detail the properties of a novel kind of beam structure that we call Whittaker-Gaussian beam and that constitute a new special case of the general CiBs. In particular we analytically determine the normalization constant, the M^2 factor, and the kurtosis parameter of the beam for a wide range of its parameters.

2. DEFINITION OF THE CIRCULAR BEAMS

In this section, we briefly describe the CiBs in order to establish notation and to provide a reference point for necessary formulas. We refer the reader to Ref.[1] for a more detailed discussion. At a given plane z, the transverse field of a monochromatic CiB with time dependence $\exp(-i\omega t)$ and wave number k is described by either the Whittaker function $M_{\mu,\kappa}(x)$ or the confluent hypergeometric function ${}_1F_1(a,b;x)$ as follows:

$$\mathrm{CiB}_{\gamma}^{m}(\mathbf{r}_t;q_0,p_0) = \frac{M_{\mu/2,m/2}\left(P_0 r^2\right)}{\sqrt{P_0 r^2}} \exp\left(\frac{Q_0 r^2}{2}\right) \exp\left(im\theta\right), \tag{1a}$$

$$= \left(P_0 r^2\right)^{m/2} \exp\left(\frac{ikr^2}{2q_0}\right) {}_1F_1\left(\beta, m+1; P_0 r^2\right) \exp\left(im\theta\right), \tag{1b}$$

where $\mathbf{r}_t = (x,y) = (r,\theta)$ denotes the transverse coordinates and the quantities β, P and Q are defined by

$$\beta \equiv \frac{m+1-\mu}{2}, \tag{2}$$

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$$P_0 \equiv \frac{ik}{2}\left(\frac{1}{p_0}-\frac{1}{q_0}\right), \qquad Q \equiv \frac{ik}{2}\left(\frac{1}{p_0}+\frac{1}{q_0}\right). \tag{3}$$

The shape of the CiBs is characterized by four parameters: The parameters $\mu \in \mathbb{C}$ and $m = 0, 1, 2, \ldots$ play the role of a complex continuous radial order and an integer angular mode number, respectively. The parameters q_0 and p_0 are the values of two independent complex beam parameters that control the properties of two complex Gaussian apodizations and directly affects the physical size of the beam at the plane z. The square integrability across the whole transverse plane of the CiBs [i.e. the finiteness of the power integral $\int\int |\mathrm{CiB}|^2 \, \mathrm{d}x\mathrm{d}y$] is ensured by satisfying the conditions

$$\mathrm{Im}(1/q_0) > 0, \qquad \mathrm{Im}(1/p_0) > 0. \tag{4}$$

Let us now consider that wave function described by Eq. (1a) is going to be propagated throught a paraxial ABCD system. For an input field with parameters (γ, m, q_0, p_0), the field at the output plane of the system is given by

$$\mathrm{CiB}_\gamma^m(\mathbf{r}_t; q, p) = \zeta \left[\frac{M_{\mu/2,m/2}\left(Pr^2\right)}{\sqrt{Pr^2}} \exp\left(\frac{Qr^2}{2}\right)\exp(im\theta)\right], \tag{5a}$$

$$= \zeta \left[\left(Pr^2\right)^{m/2} \exp\left(\frac{ikr^2}{2q}\right) {}_1F_1\left(\beta, m+1; Pr^2\right)\exp(im\theta)\right], \tag{5b}$$

where the transformation laws for the beam parameters from the input plane to the output plane are

$$q = \frac{Aq_0+B}{Cq_0+D}, \qquad p_{out} = \frac{Ap_0+B}{Cp_0+D}, \tag{6}$$

and

$$\zeta \equiv \frac{\left(A+B/p_0\right)^{(\mu-1)/2}}{\left(A+B/q_0\right)^{(\mu+1)/2}}. \tag{7}$$

is an overall amplitude factor arising from the propagation of the beam throught the particular ABCD system.

It is important to note that, apart from the complex factor ζ, the propagated field has the same mathematical structure as the input field but evaluated with the new parameters (μ, m, q, p); thus the mathematical form of the CiBs is invariant under paraxial optical transformations. Finally, we remark that the free space propagation of the CiB along a distance z can be directly obtained from Eqs. (1a)-(7) replacing the values $[A, B; C, D] = [1, z; 0, 1]$.

3. THE WHITTAKER-GAUSS BEAM

In this work, we analyze the physical properties of the novel beam structure which is obtained by setting the input parameters (p_0, q_0) as[1]:

$$p_0 = \frac{k}{(i2/\omega_0^2 + 1/2W_0^2)}, \tag{8}$$

$$q_0 = -p_0^*. \tag{9}$$

where ω_0 and W_0 are assumed to be real and positive. For these values of p_0 and q_0, the CiB at $z = 0$ becomes

$$\mathrm{WGB}_\mu^m(\mathbf{r}_t) = \left[\left(\frac{ir^2}{2W_0^2}\right)^{-1/2} M_{\mu/2,m/2}\left(\frac{ir^2}{2W_0^2}\right)\right]\exp\left(-r^2/\omega_0^2\right)\exp\left(im\theta\right), \tag{10}$$

The expression in brackets has a constant phase, and its scale is controlled by $W_0 \in \mathbb{R}$. We call such structure a Whittaker-Gaussian beam (WGB). The WGB_μ^m carry finite power and form a biorthogonal set of solutions of the PWE. It can be shown for $\mu = 0$ that the WGBs reduce to the QBG beams introduced by Caron and Potvliege.[3] We start by discussing the transverse intensity distribution of the WGB at an initial plane for different values of m and μ. Then by calculating the moments of the beam, we completely analyze its physical

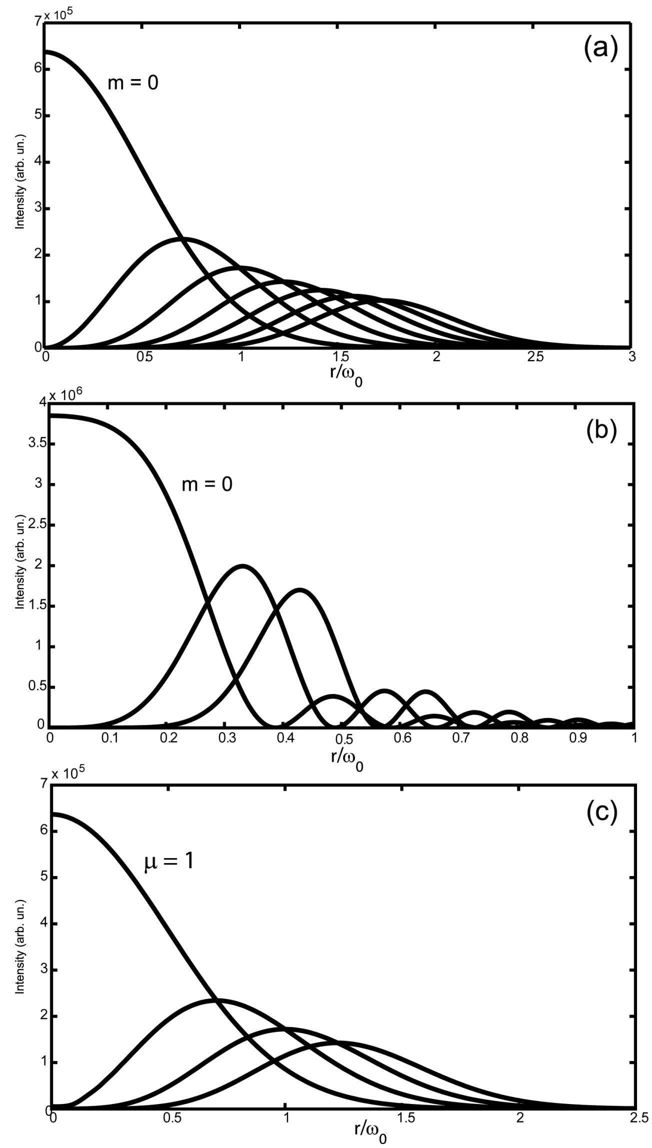

Figure 1. Transverse intensity profile of WGB beams in the plane $z = 0$ for (a) $\mathrm{Re}\,\mu = 0$ and $m = [0, 1, 2, 3, 4, 5, 6]$ (from left to right) when $\xi << 1$. For the case $\xi >> 1$ we have (b) $\mathrm{Re}\,\mu = 0$ and $m = [0, 2, 4]$ (from left to right); (c) $m = 0$ and $\mathrm{Re}\,\mu = [1, 2, 3, 4]$ (from left to right).

behavior. The normalization factor of the beam is obtained by means of the zeroth-order intensity moment σ^0. The beamwidth is related to the second-order moment σ^2, which is derived and analyzed through free-space propagation. The second-order intensity moment of the spectrum $\tilde{\sigma}^2$ in addition with σ^2 gives the beam propagation factor M^2, which relates the near-field beam size with the far-field spread of the beam. Finally, with the fourth-order moment σ^4 together with σ^2 the kurtosis parameter is calculated, which gives information about the degree of flatness (sharpness) of the beam intensity distribution.

3.1. Intensity profiles

The intensity transverse profile of a WGB for different values of its parameters m and μ is presented in Figure 1. The values of ω_0 and W_0 play an important factor in the behavior of the beam. Let's define the factor $\xi = \omega_0/W_0$. When the values of $\xi << 1$, the beam transverse intensity behaves as a simple Gaussian beam for $m = 0$. The beam presents a doughnutlike structure for values of $m > 0$ and its overall size increases as m increases. The parameter μ slightly change the intensity distribution for this case. Figure 1a shows examples for these parameters. When $\xi >> 1$ we have for $\mathrm{Re}\,\mu = 0$ and $m = 0$ a flat intensity profile at the center of the beam followed by several rings which the number of them increase with ξ. Figure 1b shows this case for $m = [0, 1, 2]$ where for higher values of m the doughnut shape appears again but preserving the number of rings. When $\mathrm{Re}\,\mu > 0$ the small rings disappear and for any value of m a doughnut shape is observed and its size control by μ. Figure 1c shows this case.

3.2. Normalization factor

The power carried by the beam corresponds to the zeroth-order intensity moment σ^0 and is calculated by the integral

$$\sigma^0_{m,\mu} = \int_0^{2\pi} \int_0^{\infty} \left|\mathrm{WGB}^m_\mu \left(\mathbf{r}_t\right)\right|^2 r \mathrm{d}r \mathrm{d}\theta. \tag{11}$$

By substituting $\mathrm{WGB}^m_\mu \left(\mathbf{r}_t\right)$ from Eq. (1a) into Eq. (11) and rearranging terms we get

$$\sigma^0_{m,\mu} = \frac{2\pi \left|\zeta\right|^2}{\left|P\right|} \int_0^{\infty} r^{-1} \exp\left(\left(\mathrm{Re}\,Q\right) r^2\right) M_{\mu/2,m/2}\left(Pr^2\right) M_{\mu^*/2,m/2}\left(P^* r^2\right) \mathrm{d}r. \tag{12}$$

The change of variables $t = r^2$ converts Eq. (12) into

$$\sigma^0_{m,\mu} = \frac{\pi \left|\zeta\right|^2}{\left|P\right|} \int_0^{\infty} t^{-1} \exp\left(\left(\mathrm{Re}\,Q\right) t\right) M_{\mu/2,m/2}\left(Pt\right) M_{\mu^*/2,m/2}\left(P^* t\right) \mathrm{d}t, \tag{13}$$

Using Eq. 7.622.3 from Gradshteyn[2] the value of $\sigma^0_{m,\mu}$ turns out to be

$$\sigma^0_{m,\mu} = \pi m! \left|\zeta\right|^2 S^{-1} \left|P/S\right|^m F_2\left(m+1; \frac{1+m-\mu}{2}, \frac{1+m-\mu^*}{2}; m+1, m+1; \frac{P}{S}, \frac{P^*}{S}\right), \tag{14}$$

where $S = \mathrm{Re}\,P - \mathrm{Re}\,Q = k\,\mathrm{Im}\left(1/q\right)$ is a real and positive number and $F_2\left(a; b, b'; c, c'; x, y\right)$ is the Appell hypergeometric function of two variables.[2,4] Using the series expansion of the F_2 Appell hypergeometric function[2] it can be demonstrated that the evaluation of Eq. (14) yields always a positive real value.

Now, the use of identity 9.182.3 of Gradshteyn[2] allows us to rewrite Eq. (14) in terms of the Gauss Hypergeometric function ${}_2F_1 = F\left(a, b; c; x\right)$ as follows:

$$\sigma^0_{m,\mu} = \frac{\pi m! \left|\zeta\right|^2}{S} \left|\frac{\left(P/S\right)^m}{\left(1 - P/S\right)^{1+m-\mu}}\right| F\left(\frac{1+m-\mu}{2}, \frac{1+m-\mu^*}{2}; m+1; \left|\frac{P/S}{\left(1-P/S\right)}\right|^2\right). \tag{15}$$

Replacing P and ζ from Eqs. (3) and (7) into Eq. (15), we can verify that $\sigma^0_{m,\mu}$ is an invariant quantity for beam propagation through any paraxial system characterized by unimodular ABCD matrices (i.e. $AD - BC = 1$) with real elements. Therefore, we can set $[A, B; C, D] = [1, 0; 0, 1]$ and substitute $[p_0, q_0]$ from Eqs. (8)-(9) into Eq. (15) to obtain:

$$\sigma^0_{m,\mu} = \frac{\pi m! \omega_0^2}{2^{2m+1}} \frac{\xi^{2m}}{\left|\left(1 - i\xi^2/4\right)^{1+m-\mu}\right|} F\left(\frac{1+m-\mu}{2}, \frac{1+m-\mu^*}{2}; m+1; \frac{\xi^4}{16+\xi^4}\right), \tag{16}$$

where $\xi = \omega_0/W_0$ as before. Once calculated the power $\sigma^0_{m,\mu}$, the normalization constant of WGB^m_μ is determined as $C_{m,\mu} = 1/\sqrt{\sigma^0_{m,\mu}}$.

3.3. Beam quality factor M^2

The beam quality factor M^2 is a common parameter characterizing the propagation features of a light beam.[?] In general, for a fixed width at the waist plane a better-quality beam is associated with a lower value of M^2. A minimum value of 1 is reached only for the fundamental Gaussian beam. For rotationally symmetric beams, the M^2 factor is given by[5]

$$M^2 = \sqrt{\left(\frac{\sigma_{waist}^2}{\sigma^0}\right)\left(\frac{\tilde{\sigma}^2}{\sigma^0}\right)}, \tag{17}$$

where σ_{waist}^2 and $\tilde{\sigma}^2$ are the second-order intensity moments at the waist plane and in the far field, respectively.

The second-order intensity moment σ^2 is

$$\sigma_{m,\mu}^2 = \int_0^{2\pi}\int_0^{\infty} r^2 \left|\mathrm{WGB}_\mu^m\right|^2 r\mathrm{d}r\mathrm{d}\theta, \tag{18}$$

and can be calculated by means of the same procedure given for the zeroth-order intensity moment. However, can be noticed in Eq. (12) that the derivative of $\sigma_{m,\mu}^0$ with respect to $\mathrm{Re}\,[Q]$ yield to $\sigma_{m,\mu}^2$:

$$\sigma_{m,\mu}^2 = G_\mu^m \sigma_{m,\mu}^0 + H_\mu^m \sigma_{m+1,\mu-1}^0, \tag{19}$$

where the factors G_μ^m and H_μ^m are given by:

$$G_\mu^m = \mathrm{Re}\left[\frac{m+1}{S} + \left(\frac{m+1-\mu}{S}\right)\left(\frac{P}{S-P}\right)\right], \tag{20}$$

$$H_\mu^m = \mathrm{Re}\left[\frac{|m+1-\mu|^2}{2\,(m+1)^2}\frac{(S-P)}{S}\frac{1}{|P|}\left|\frac{P_0}{S_0-P_0}\right|^2\left|\frac{A+B/p_0}{A+B/q_0}\right|\right], \tag{21}$$

and $\sigma_{m+1,\mu-1}^0$ is the zero-order moment of the shifted beam $\mathrm{WGB}_{\mu-1}^{m+1}$.

It is well known that for free space propagation, the second order moment has a parabolic function of the distance from the waist plane and it is related to $\tilde{\sigma}^2$ through the equation[5] $\sigma^2 = \sigma_{waist}^2 + \left(\tilde{\sigma}^2/k^2\right)(z - z_{waist})^2$. Using Eqs. (19)-(21) with $[A, B; C, D] = [1, z; 0, 1]$ and rearranging terms:

$$\frac{\sigma_{m,\mu}^2}{\sigma_{m,\mu}^0} = \frac{W_0^2\left(u_{m,\mu}^2 - \xi^4\,(\mathrm{Re}\,\mu)^2 - 16\,|\mu|^2\right)}{2\left(1+16\xi^{-4}\right)(u_{m,\mu} - 4\,\mathrm{Im}\,\mu)} + \frac{u_{m,\mu} - 4\,\mathrm{Im}\,\mu}{8k^2W_0^2}\left(z - \frac{4z_R\,\mathrm{Re}\,\mu}{u_{m,\mu} - 4\,\mathrm{Im}\,\mu}\right)^2, \tag{22}$$

where $z_R = k\omega_0^2/2$ is the Rayleigh distance and ξ was defined previously. Therefore, from Eq. (22) the expression for z_{waist}, σ_{waist}^2 and $\tilde{\sigma}^2$ are:

$$z_{waist} = \frac{4z_R\,\mathrm{Re}\,\mu}{u_{m,\mu} - 4\,\mathrm{Im}\,\mu}, \tag{23}$$

$$\frac{\sigma_{waist}^2}{\sigma_{m,\mu}^0} = \frac{W_0^2\left(u_{m,\mu}^2 - \xi^4\,(\mathrm{Re}\,\mu)^2 - 16\,|\mu|^2\right)}{2\left(1+16\xi^{-4}\right)(u_{m,\mu} - 4\,\mathrm{Im}\,\mu)}, \tag{24}$$

$$\frac{\tilde{\sigma}^2}{\sigma_{m,\mu}^0} = \frac{u_{m,\mu} - 4\,\mathrm{Im}\,\mu}{8W_0^2} \tag{25}$$

and u is given by

$$u_{m,\mu} = \frac{2\,|1+m-\mu|^2}{(1+m)^2}\frac{\sigma_{m+1,\mu-1}^0}{\sigma_{m,\mu}^0} + 16\xi^{-2}\,(1+m) + \xi^2\,\mathrm{Re}\,(\mu)\,. \tag{26}$$

The beamwidth is related to the second-order moment by $W^2(z) = 2\sigma^2$. By Eq. (23) is observed that the waist position is located at the plane $z = 0$ when μ is purely imaginary. The real value of μ displace the position of the waist. Figure 2 shows the propagation of the beamwidth for different values of m and μ.

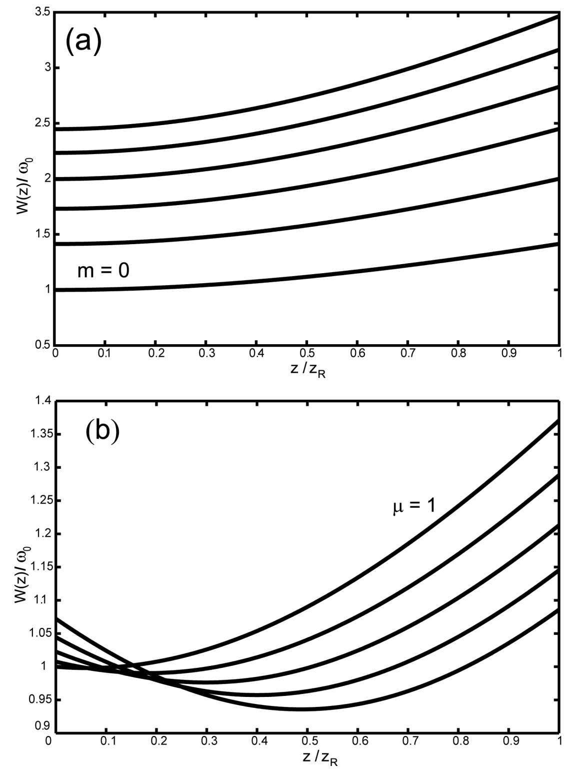

Figure 2. Normalized beamwidth propagation of WGBs for (a) $\mathrm{Re}\,\mu = 0$ and values of $m = [0, 1, 2, 3, 4, 5]$ (from down to top); (b) $m = 0$ and $\mathrm{Re}\,\mu = [1, 3, 5, 7, 9]$ (from top to down). The waist position changes with the real part of μ.

The M^2 factor is now calculated using Eqs. (22)-(25) with Eq. (17). The result is

$$M^2_{m,\mu}(\xi) = \frac{1}{4}\sqrt{\frac{u^2_{m,\mu} - \xi^4 (\mathrm{Re}\,\mu)^2 - 16\,|\mu|^2}{1 + 16\xi^{-4}}}. \tag{27}$$

It is noted that M^2 depends on the relation between ω_0 and W_0, meaning ξ. Figure 3 shows the M^2 factor with respect to the parameter ξ. The initial value of M^2 (when $\xi \to 0$) is always given by $m + 1$. When increasing the parameter ξ, we have two situations, for values where $0 \leq \mathrm{Re}\,\mu \leq 1$ the M^2 factor tends to infinity for each value of m. When $\mathrm{Re}\,\mu > 1$ the M^2 tends to a constant value given by the expression:

$$M^2_{m,\mu}(\xi \to \infty) = \sqrt{|1 + m - \mu|^2\,\Gamma(\mathrm{Re}\,[\mu - 1])\,\mathrm{Re}\,(\mu)\,/\,\Gamma(\mathrm{Re}\,\mu) + 2\,(1 + m)\,\mathrm{Re}\,(\mu) - |\mu|^2}, \tag{28}$$

from which can be easily shown that for the special cases when $\mu = m + 1$, the beam propagation factor remains constant to the initial value $\mathrm{M}^2 = m + 1$.

3.4. Kurtosis parameter

The kurtosis parameter is defined as[6]

$$\mathcal{K} = \frac{(\sigma^4/\sigma^0)}{(\sigma^2/\sigma^0)^2} = \frac{\sigma^0\sigma^4}{(\sigma^2)^2}, \tag{29}$$

where σ^4 corresponds to the fourth-order intensity moment and describes the degree of flatness (or sharpness) of the beam intensity distribution. The beam profile is classified as leptokurtic (sharper profiles), mesokurtic, or platykurtic (flatter profiles) depending on $\mathcal{K}$ being larger, equal or less than 2, which is the kurtosis value of the

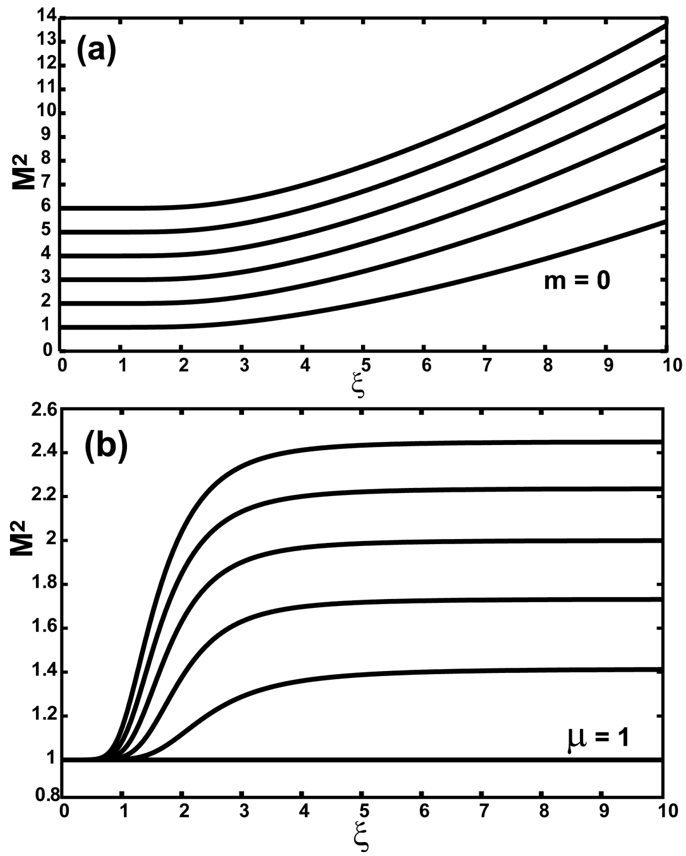

Figure 3. Beam propagation factor of WGB versus ξ when (a) $0 \leq \operatorname{Re}\mu \leq 1$ and $m = [0, 1, 2, 3, 4, 5]$ (from down to top) (b) $m = 0$ for values of $\mu = [1, 2, 3, 4, 5, 6]$ (from down to top).

axially symmetric Gaussian beam. While M^2 is an invariant quantity for beam propagation through any ABCD optical system, $\mathcal{K}$ changes on the propagation except for some special cases.

By repeating the strategy to find σ^2, we have $\sigma^4 = [\mathrm{d}/\mathrm{d}(\operatorname{Re} Q)]\,\sigma^2$ and using Eq. (19) the result is:

$$\sigma^4_{m,\mu} = \left\{ \frac{1}{S} G^m_\mu + \left(G^m_\mu\right)^2 + \frac{1}{S} \operatorname{Re}\left[\frac{(m+1-\mu)\,P}{(S-P)^2} \right] \right\} \sigma^0_{m,\mu}$$
$$+ \left[2H^m_\mu G^m_\mu + 2H^m_\mu \operatorname{Re}\left(\frac{1}{S-P} \right) - \frac{1}{S} \frac{|m+1-\mu|^2}{2\,(m+1)^2} \frac{1}{|P|} \left| \frac{P_0}{S_0 - P_0} \right|^2 \left| \frac{A + B/p_0}{A + B/q_0} \right| \right] \sigma^0_{m+1,\mu-1}$$
$$+ \left(H^m_\mu H^{m+1}_{\mu-1} \right) \sigma^0_{m+2,\mu-2}. \quad (30)$$

Numerical calculations of the kurtosis parameter are shown in Fig. 4. According to Herrero,[6] we can classify the WGBs according to the kurtosis behavior under free-space propagation. For cases when $\operatorname{Re}\mu = 0$ and $m \geq 0$ the kurtosis parameter presents two maxima and one minimum, corresponding to type III (Figure 4a). For values of $\operatorname{Re}\mu > 0$ and $m \geq 0$ there are two maxima and two minima that corresponds to type I (Figure 4b). For the special case when $\operatorname{Re}\mu = m + 1$, the kurtosis parameters becomes constant through propagation, with a value of $K = 1 + 1/\operatorname{Re}\mu$, this case corresponds to type VII. The asymptotic value of the kurtosis at the far field, $K(z \to \pm\infty) = K_\infty$, is represented by the dashed line and can be obtained by propagating the beam through a classical $2f$ optical system characterized by the ABCD matrix $[0, k; -1/k, 0]$.

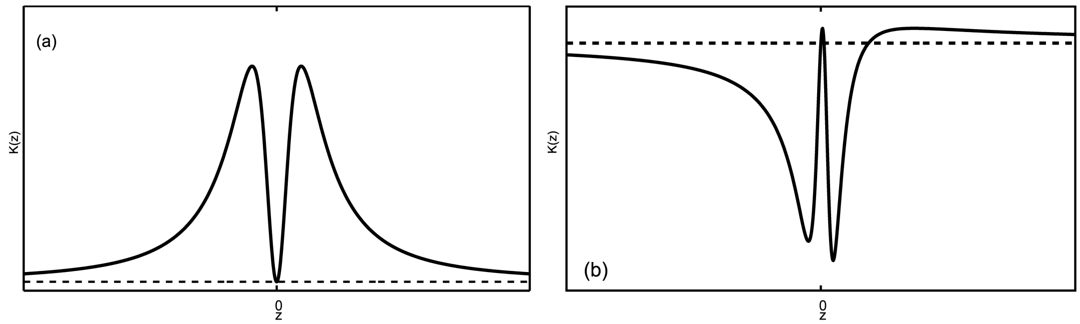

Figure 4. Free-space propagation of kurtosis parameter. There are two cases when (a) $\mathrm{Re}\,\mu = 0$ and $m \geq 0$ and (b) $\mathrm{Re}\,\mu > 0$ and $m \geq 0$. For the special case when $\mathrm{Re}\mu = m+1$ the kurtosis parameter becomes constant to $K = 1+1/\,\mathrm{Re}\,\mu$.

4. CONCLUSIONS

We have studied in detailed the physical properties of the novel Whittaker-Gaussian Beams. Important propagation factors were obtained through the 2n-order intensity moments of the beam. Analytical expressions for the normalization factor, beamwidth, beam propagation factor M^2 and kurtosis parameter K were obtained and numerical calculations performed for different values of the parameters.

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